Why Sunk Costs Matter for Bargaining Outcomes: An Evolutionary Approach

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Two bargaining parties play the Nash Demand Game to share a pie whose size is determined by one party's investment decision. Various investment levels are subgame-perfect. Adding the investment decision to Young's evolutionary bargaining model yields the following long-run outcome: efficient investment prevails and the investor's share of the pie approximates the maximum of (i) the smallest share that induces efficient investment, even if the investor expects to appropriate the available pie from every inefficient investment, and (ii) half of the pie. The result favors forward induction to subgame consistency and equity theory to hold-ups. Journal of Economic Literature Classification Numbers: C78, L14.

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1. INTRODUCTION

When two parties bargain over their shares of a pie, it is often the case that one of them has sunk costs in advance, and the size of the pie depends on these costs. How will the shares of the pie depend on the sunk costs? What incentives to sink costs will result? To tackle these questions, we define a two-stage game, the Investment Demand Game. In stage 1, player 1 can incur costs in order to increase the value of a pie to be shared in stage 2. In stage 2, player 1 and 2 bargain over their shares of the pie. The bargaining model is the Nash Demand Game: both parties simultaneously...
demand shares of the available pie; if the demands sum up to more than the value of the pie, it is lost; otherwise, each player obtains her demand.

The reason why we use the Nash Demand Game is that in this bargaining model all splits of the pie are Nash equilibria. I.e., Nash equilibrium implies no restriction on the bargaining outcome, and in this sense the potential effect of sunk costs on the bargaining outcome is unrestricted, too. As a result, there are generally many subgame-perfect investment levels in the Investment Demand Game.

The motivation of this paper is to shed light on a tension between the theory of incomplete contracts and the psychological equity theory. The former theory emphasizes that contractual provisions are necessary “so that investing parties do not fear that subsequent bargaining will “rob” them of the value of their specific investment” (Rogerson, [15, p. 777]). This suggests an equilibrium with inefficiently low investment in the Investment Demand Game. Equity theory, however, “shows that people feel that those who have put more effort into creating resources have more claim on those resources,” (Rabin [14, p. 18]) which suggests an equilibrium with enhanced investment incentives.

Which of these suggestions is more “appropriate?” We give an answer based on an evolutionary process in which the Investment Demand Game is played repeatedly. It turns out that, under suitable technical assumptions, in the long-run average a very narrow set of equilibrium outcomes occurs much more often than any other feasible outcome. This set of long-run outcomes is our prediction for behavior.

The evolutionary process is an extension of Young's [25] model. In each period, the Investment Demand game is played by two agents who are randomly drawn from large populations. Agents are boundedly rational in three respects: they use their information about their environment in a naive way, they act myopically, and sometimes they make random “experiments.” An agent’s information is (separately for each investment level) a random sample of demands made by opponent agents in past periods. She derives a conjecture about the opponent’s behavior from her sample, by calculating a frequency distribution over opponents’ demands following each feasible investment level. Unless the agent makes an experiment, she plays a myopic best reply to her conjecture. Experiments are modelled as realizations of an exogeneous random variable.

Without experiments, the steady-state outcomes of the process would coincide with the subgame-perfect equilibrium outcomes. I.e., without experiments evolution would leave us with the same predictive indeterminacy as subgame-perfect equilibrium. However, the possibility of experiments provides a perturbation which from time to time triggers a sequence

\footnote{For an exposition of this theory in an economic context, see Selten [17] or Binmore [3].}
of transitions leading from one equilibrium outcome to another. The long-run outcomes are those equilibrium outcomes which (in a sense we will make precise) are most stable against experiments.

Our main result is that in all long-run outcomes the level of sunk costs is efficient and, given that all agents sample the same number of past demands, player 1’s share of the efficient pie approximates the maximum of (i) the smallest share that induces efficient investment, even if player 1 expects to appropriate the available pie from any inefficient investment; (ii) half of the efficient pie.

For instance, suppose there exist two investment possibilities. One is costless and creates a pie of value 8, the other creates a pie of value 20 at cost 5. In the long run, the pie 20 is created, and player 1’s share of it is approximately $8 + 5 = 13$ (which is more than half of the pie’s value). If the costs were not 5 but only 1, then player 1’s share would approximate $20/2 = 10$.

The result demonstrates that sunk costs matter for bargaining outcomes in a comparative-static sense: the set of long-run outcomes depends on the investment cost function. Moreover, sunk costs matter in such a way that efficient investment incentives are provided. Hence, evolution overcomes the hold-up problem which, according to incomplete contracts theory, tends to occur when bargaining follows investment in the absence of contractual provisions. We emphasize that this result is obtained although equity or fairness considerations are not part of the players’ behavior rule.

Under assumptions very similar to ours, Young [25] shows that the long-run outcomes of the Nash Demand Game (in the absence of investment possibilities) approximate the fifty-fifty split of the pie. Combining the main result of [25] with our paper yields an evolutionary violation of Harsanyi and Selten’s [8] principle of subgame consistency: there exists no subgame consistent solution concept that, for each Nash Demand Game and each Investment Demand Game, selects an equilibrium whose outcome is a long-run outcome.

Our result is also related to another equilibrium selection principle: strategic stability (Kohlberg and Mertens [10, Section 3.5]). We show that all long-run outcomes are strategically stable. Thus, evolution supports the forward induction intuition incorporated in the notion of strategic stability. This is similar to Nöldeke and Samuelson’s [12] result that, in a related evolutionary model, a forward induction property holds for the long-run outcomes of a class of normal form games with an outside option.

Our evolutionary process extends Young’s [25] evolutionary bargaining model to the Investment Demand Game as follows: when not experimenting, each player applies Young’s [25] behavior rule to each stage-2 subgame; player 1 performs one step of backward induction to determine her optimal investment level; if player 1 experiments, she chooses a random
investment and a random demand; if player 2 experiments, she chooses a random demand.

Technically, our evolutionary process is a “regular perturbation”-family of Markov processes in the spirit of Kandori et al. [9], Young [24], and many successors. To calculate the long-run outcome of the Investment Demand Game, we will however not directly apply the mutation-counting technique of [9] and [24]. This would be quite tedious because of the huge number of equilibria. It is easier to first follow Nöldeke and Samuelson’s [12] “single-mutation” technique which provides a necessary condition for equilibrium outcomes to be long-run outcomes. Applied to the Investment Demand Game, Nöldeke and Samuelson’s technique shows that all long-run outcomes include efficient investment, and player 1’s share of the efficient pie is at least that described in (i) above. To actually show that player 1’s share approximates the maximum of (i) and (ii), we have to extend Nöldeke and Samuelson’s [12] technique, thereby combining it with Young’s [24] technique. We do this by a general technical result which can be applied to every regular perturbation.

In Section 2 we define the Investment Demand Game and discuss the above mentioned equilibrium selection devices. Section 3 defines the evolutionary process. Section 4 presents the result which describes the long-run outcomes of the Investment Demand Game. We also sketch several extensions, and compare the long-run outcomes to a standard of equity theory. In Section 5, we discuss some related literature. Section 6 contains two appendices. In Appendix A, we state and prove the general technical result mentioned above. In Appendix B, we prove the result stated in Section 4.

2. THE INVESTMENT DEMAND GAME

In the Nash [11] Demand Game two players, called 1 and 2, bargain over a pie \( v > 0 \) by simultaneously announcing demands \( x, y > 0 \). The players’ payoffs are

\[
(u_1(x, y), u_2(x, y)) = \begin{cases} 
(x, y), & \text{if } x + y \leq v, \\
(0, 0), & \text{if } x + y > v.
\end{cases}
\]

We say that \( v \) is realized (/lost) if \( u_1(x, y) + u_2(x, y) = v (/= 0) \). All pairs \( (x, v - x) \) with \( x \in (0, v) \) are Nash equilibria in which \( v \) is realized for sure. There are also pure strategy Nash equilibria in which \( v \) is lost for sure (for instance, \( (v, 0) \)), and many mixed strategy Nash equilibria in all of which \( v \) is lost with positive probability.
The Nash Demand Game will now be embedded in a two-stage game: in the first stage player 1 invests $C(v)$. In the second stage (after $C(v)$ is sunk) players 1 and 2 play the Nash Demand Game with pie $v$. The payoffs of this **Investment Demand Game** are

$$
(u_1(v, x, y), u_2(v, x, y)) = \begin{cases} 
(x - C(v), y), & \text{if } x + y \leq v, \\
(-C(v), 0), & \text{if } x + y > v.
\end{cases}
$$

The subgame following investment $C(v)$ is called the $v$-subgame.

The Investment Demand Game models bargaining with sunk costs. The specification is “simple” in several respects: only two parties are involved, only one of them can invest, there is no uncertainty about the value of the pie for any given investment level, and the bargaining stops after one round of simultaneous demands.

Player 1’s action in stage 1 can be interpreted as any activity that incurs sunk costs and creates a bilateral monopoly. For instance, extending Young’s [25] running example, player 1 might be a tenant, and player 2 might be a landlord. They bargain over their shares of the crop by simultaneously announcing demands $x$ and $y$. The value $v$ of the crop depends on the tenant’s effort costs $C(v)$ (note that, for presentational reasons, the tenants represent player 1, whereas in [25] they represent player 2).

To facilitate the application of the theory of finite Markov processes, we specify finite action sets. We assume that there exists a finite set $\mathcal{V}$ of **feasible pies**, $v \in \mathcal{V} = \{v_1, \ldots, v_{|\mathcal{V}|}\}$, and a **base unit** $\delta > 0$ such that $v_i/\delta$ ($i = 1, \ldots, |\mathcal{V}|$) are integers and $v_i/\delta \geq 2$.$^4$ We assume that

$$x, y \in (0, v] \cap \{\delta, 2\delta, \ldots\}$$

if $x$ and $y$ are demands for pie $v$. A triple $(v, x, y)$ which is consistent with all these restrictions is called a **feasible outcome**. Notice that we have forbidden all demands larger than $v$. This is without loss of generality because these demands are payoff-equivalent to the demand $v$. We also assume that player 1 is not indifferent between any two different feasible outcomes $(v, x, v - x)$ and $(v', x', v' - x')$. Note that this holds for generic cost functions.

$^4$ The last assumption is made for presentational convenience only. It assures for every $v$-subgame the existence of a Nash equilibrium $(v, -\delta, \delta)$ with surely realized pie.
If provision of some pie $v$ can be supported by any Nash equilibrium, then $v$ can also be supported by a Nash equilibrium with the property that the demand profile $(v', v')$ is played in all $v'$-subgames ($v' \neq v$). Therefore, a feasible outcome $(v, x, y)$ is a Nash equilibrium outcome with surely realized pie if and only if

$$x + y = v \quad \text{and} \quad x - C(v) \geq \max_{v' \not\in v \setminus \{v\}} C(v').$$

The efficient pie

$$v^* = \arg \max_{v \in \mathcal{V}} (v - C(v)).$$

is always an equilibrium pie, but generally many more equilibrium pies exist. The equilibrium multiplicity is not resolved by subgame-perfection: every Nash equilibrium outcome is a subgame-perfect equilibrium outcome.

The evolutionary analysis started in Section 3 will provide an equilibrium selection device. As a benchmark, we now briefly discuss two “rational” equilibrium selection devices: subgame consistency and strategic stability. Supposing that for each game (in a given set of games) one Nash equilibrium is selected as the “solution,” the principle of subgame consistency (Harsanyi and Selten [8]) says that the behavior a game’s solution prescribes in any of its subgames equals the solution to that subgame. According to this principle, the demands in any given subgame of the Investment Demand Game cannot (in a comparative-static sense) depend on the cost function $C(\cdot)$. In this sense, subgame consistency implies that sunk costs do not matter for bargaining outcomes. If the principle of subgame consistency is combined with selecting for the Nash Demand Game an equilibrium with an approximate fifty-fifty split, the conclusion is that, under standard assumptions on the cost function $C(v)$, investment will be inefficiently low.

This kind of inefficiency is known as the hold-up problem in the context of firms investing in trading relationships with incomplete contracting (see Tirole [21] for a bibliography of this strand of literature). The hold-up problem plays a prominent role in the theory of incomplete contracts.

A second set of equilibrium selection devices relevant for the Investment Demand Game goes under the heading of “forward induction.” We confine ourselves to Kohlberg and Mertens’s [10, Section 3.5] definition of strategic stability which captures some of the intuition behind forward induction.\footnote{Van Damme [23] criticizes [10]. An extensive discussion of forward induction can be found in Fudenberg and Tirole [7, Chapter 11].}
The remark below shows that strategic stability differs remarkably from subgame consistency. In fact, all strategically stable outcomes\(^6\) include efficient investment; moreover, player 1’s share of the efficient pie is so large that she would invest efficiently even if she appropriated (up to the base unit \(\delta\)) the entire pie obtained from any inefficient investment.

Remark 2.1. Assume \(|V'| \geq 2\).\(^7\) A feasible outcome \((v, x, v-x)\) is strategically stable if and only if \(v = v_*\) and \(x > x_* - \delta\), where

\[
x_* = C(v_*) + \max_{v' \in V \setminus \{v_*\}} \left( v' - C(v') \right).
\]

The proof of the “if”-part (the details are omitted) utilizes that, in the reduced normal form, the player-1 strategy \((v_*, x)\) (with \(x > x_* - \delta\)) is a strict best reply to any strategy in which player 2 demands \(v_* - x\) in the \(v_*\)-subgame, and \(v_* - x\) is a strict best reply demand of player 2 against \(x\) in the \(v_*\)-subgame. Hence, the set of equilibria with outcome \((v_*, x, v_* - x)\) is stable with respect to small trembles of the strategies in the sense required by strategic stability, and therefore this set has a strategically stable subset.\(^8\)

The proof of the “only if”-part (the details are omitted) utilizes two properties of strategic stability. First, any strategically stable outcome \(o\) remains strategically stable in the game obtained by deletion of those strategies which are never weak best replies to any strategy in a Nash equilibrium with outcome \(o\). Second, \(o\) remains strategically stable if weakly dominated strategies are deleted. In the Investment Demand Game, no outcome \((v, x, v-x)\) with \(v \neq v_*\) or \(x < x_* - \delta\) can be strategically stable because after deletion of the player-1 strategies which are never weak best replies to any strategy in a Nash equilibrium with outcome \((v, x, v-x)\), all player-2 strategies with sufficiently large demands get weakly dominated, implying that player 1 has a strategy which gives her a certain payoff which is higher than \(x - C(v)\).

Remark 2.1 shows that according to strategic stability, sunk costs matter for bargaining outcomes in a comparative-static sense: the set of strategically stable equilibrium outcomes depends on the cost function \(C(\cdot)\).

However, strategic stability does not completely answer the question how

\(\text{\footnotesize \cite[Theorem 1]{Swinkels19}}\) yields an alternative proof of the “if”-part.

\(^6\) We say that an outcome \(o\) has a property (e.g., strategy stability) if there exists a set of strategy profiles with this property and with outcome \(o\).

\(^7\) If \(|V'| = 1\), any equilibrium in which the pie is surely realized is strict and hence strategically stable.

\(^8\) One can show that a Nash equilibrium outcome \((v, x, v-x)\) of the Investment Demand Game is equilibrium evolutionary stable (Swinkels \cite{Swinkels18}) and uniformly robust to equilibrium entrants (Swinkels \cite{Swinkels19}), if and only if \(v = v_*\) and \(x > x_* - \delta\). Hence, Swinkels \cite[Theorem 1]{Swinkels19} yields an alternative proof of the “if”-part.
sunk costs matter, because a possibly broad set of pie shares is consistent with strategic stability.

The principle of subgame consistency selects among equilibria, while strategic stability selects among equilibrium outcomes. This latter property is shared by the evolutionary process defined below. Moreover, evolution will—under appropriate technical assumptions—select a very narrow subset of the strategically stable outcomes.

3. POPULATIONS AND EVOLUTION

The following definition of an evolutionary process extends Young’s [25]. Player 1 is represented by a finite population $A$ of agents, which are called the tenants. Likewise, player 2 is represented by a finite population $B$ of landlords. In each period $t = 1, 2, \ldots$, a pair of agents $(\pi_t, \beta_t) \in A \times B$ is randomly drawn to play the Investment Demand Game. These agents rely on records of (truncated) histories of all the stage-2 subgames. More precisely, for every $v \in \mathcal{V}$, we define the record

$$s_t(v) = ((x_1^1(v), y_1^1(v)), \ldots, (x_m^v(v), y_m^v(v))),$$

where $(x_i^v(v), y_i^v(v))$ denotes the demands in the $i$th recent play of the $v$-subgame. The record $s_t(v)$ is an example of a $v$-memory, a vector of $m$ feasible demand pairs in the $v$-subgame. Any collection of $v$-memories $(v \in \mathcal{V})$ is called a state. The set of feasible states is denoted $\Theta$. At the beginning of period $t$, the process is in state $s_t = (s_t(v))_{v \in \mathcal{V}}$. The state $s_1$ at the beginning of period 1 is called the initial state.

The agent $\pi_t$ starts with calculating for each $v \in \mathcal{V}$ what she would do if the $v$-subgame were entered. The calculation is based on a random sample drawn from $s_t(v)$. The sample consists of $k$ demands made by the landlords, and $\pi_t$ calculates a best reply demand to the frequency distribution of the demands in the sample. Having done this for all $v \in \mathcal{V}$, she derives an optimal pie $v_t^\pi$ from her expected payoffs in all stage-2 subgames. We call the resulting play $(v_t^\pi, x_t^\pi)$ a best reply to the state $s_t$, and we call $v_t^\pi$ a best reply pie in state $s_t$. In a similar way, calculating for agent $\beta_t$ a best reply to $s_t$ results in some demand $y_t^\beta$ of $\beta_t$ in the $v_t^\beta$-subgame.

Note that while there is no claim that any agent is aware of the evolutionary process he or she is a part of, all agents are aware of the structure of the Investment Demand Game they are playing. In particular, the tenants’ behavior rule implicitly assumes backward induction reasoning.

We also allow for the possibility that the agents sometimes make mistakes or experiment with strategies which are not best replies. Let $\varepsilon > 0$ be the probability that any given agent experiments in any given period. If $\pi_t$
experiments, she chooses a pair $(v_t^x, x_t^y)$ at random; otherwise, she chooses a demand $y^*_t$ at random. Therefore, the eventual period-$t$ outcome $(v_t, x_t, y_t)$ equals $(v_t^x, x_t^y, y_t^y)$ with probability $(1-\varepsilon)^2$, equals $(v_t^x, x_t^x, y_t^y)$ with probability $(1-\varepsilon)\varepsilon$, equals $(v_t^x, x_t^y, y_t^x)$ with probability $(1-\varepsilon)\varepsilon$, and equals $(v_t^y, x_t^x, y_t^y)$ with probability $\varepsilon^2$.

The outcome $(v_t, x_t, y_t)$ determines the state $s_{t+1} = (s_t(v))_{v\in V^1}$ at the beginning of period $t+1$ such that memory updating occurs on the observed path of play, while the observed play is not used to update any memory off the observed path. Formally, for all $v \neq v_t$, let $s_{t+1}(v) = s_t(v)$; for $v = v_t$, let

$$s_{t+1}(v) = ((x_t, y_t), (x_t^y(v), y_t^y(v)), ..., (x_t^{n-1}(v), y_t^{n-1}(v))). \tag{2}$$

The situation that the process is in state $s_t$ at the beginning of period $t$, and is in state $s_{t+1}$ at the beginning of period $t+1$, is called a transition from $s_t$ to $s_{t+1}$.

We assume that matching, sampling, selecting a best reply if there is more than one, and experimenting in period $t$, are described by stochastically independent probability distributions which may depend on the current state $s_t$, but are otherwise independent of the period $t$. These distributions have maximal supports (i.e., each pair of agents in $A \times B$ is drawn with positive probability, each sample of size $k$ is drawn with positive probability, each best reply is played with positive probability, and each feasible action can occur as an experiment).

For all states $s, s' \in \Theta$ (and, by construction, independently of the period $t$), the probability of transition from state $s \in \Theta$ to state $s' \in \Theta$ is denoted $P^e_{s \rightarrow s'}$. The Markov process $P^e = (P^e_{s \rightarrow s'}), s, s' \in \Theta$ is called the adaptive play process with experimentation rate $\varepsilon$. The process $P^e$ has the property that each state can be reached from each other by a sequence of $m |T^1|$ subsequent transitions, as well as by $m |T^1| + 1$ subsequent transitions (to see this, suppose that both players experiment in $m |T^1|$ consecutive periods, cf. Young [24, pp. 67–68]). Moreover, each probability $P^e_{s \rightarrow s'}$ is polynomial in $\varepsilon$. Therefore, the family of processes $(P^e), e \in (0,1)$ is a regular perturbation in the sense of Young [p. 77]. This implies the following result.

**Lemma 3.1.** The process $P^e$ has a unique stationary distribution $\mu^e = P^e \mu^e$. For every initial state $s_1$,

$$\mu^e = \lim_{t \to \infty} (P^e)^t s_1.$$
The time average of the frequency of any state \( s \in \Theta \) converges to \( \mu^*(s) \) with probability 1, independently of the initial state. Moreover, \( \mu^* = \lim_{\varepsilon \rightarrow 0} \mu^\varepsilon \) exists.

As Kandori et al. [9], Young [24], and their followers, we are not going to calculate the stationary distribution \( \mu^\varepsilon \) for every \( \varepsilon \). Instead we explore properties of \( \mu^* \). A convenient interpretation of the behavior pattern induced by the distribution \( \mu^* \) is "what happens in the long run if the experimentation rate is small."

We call a state \( s \) a long-run state if \( \mu^*(s) > 0 \). Notice that if the experimentation rate \( \varepsilon \) is sufficiently small, then according to \( \mu^\varepsilon \) all long-run states have positive probability, and the probability that the process is in one of the long-run states is arbitrarily close to one. An outcome \( o \) is called a long-run outcome if there exists a long-run state in which, conditional on the event that no experiment occurs, the outcome \( o \) occurs with positive probability.

4. THE INVESTMENT DEMAND GAME IN THE LONG RUN

In this section we characterize the long-run outcomes of the Investment Demand Game when there is more than one feasible pie. The agents are assumed to sample not more than half of the records included in the state. Moreover, given the base unit \( \delta \), the sample size \( k \) must not be too small.

**Proposition 4.1.** Assume \( |\mathcal{V}| \geq 2 \).

If, given \( \delta \), the sample size \( k \) is sufficiently large and \( m \geq 2k \), then every long-run outcome \( (v, x, y) \) has the properties

\[
v = v^*_a, \quad x - \max \left\{ x^*, \frac{v^*_a}{2} \right\} \leq \delta, \quad y = v^*_a - x,
\]

where

\[
x^*_a = C(v^*_a) + \max_{v \in \mathcal{V} \backslash \{v^*_a\}} (v - C(v)).
\]

Hence, under the assumptions of Proposition 4.1, the long run tenants' share \( x \) of the efficient pie \( v^*_a \) is such that efficient investment is induced, and \( x \) is, up to the base unit \( \delta \), the maximum of two numbers: \( x^*_a \)—the smallest share which induces efficient investment even if the tenants expect to appropriate the whole pie after any inefficient investment; \( v^*_a/2 \)—the tenants' share in the Nash Bargaining Solution.
On the one hand, Proposition 4.1 demonstrates that if there exists an inefficient investment very similar to the efficient one, i.e., if investment levels can be “fine tuned,” then the tenants appropriate almost the whole efficient pie \( v_* \). On the other hand, if all inefficient pies are sufficiently small or expensive and the efficient pie is sufficiently cheap, then the inefficient investment possibilities are “irrelevant,” and the efficient pie is shared according to the Nash Bargaining Solution.

Proposition 4.1 shows that sunk costs matter for bargaining outcomes in a comparative-static sense: which equilibrium outcomes are selected in the long run depends on the cost function \( C(\cdot) \). Moreover, the long-run outcome provides (up to the base unit \( \$ \)) a precise prediction about how sunk costs matter. However, no prediction is made about the players’ shares of the inefficient pies: our proof shows that any feasible profile of splits of the inefficient pies is represented by some long-run state. In other words, the evolutionary process selects among outcomes rather than selecting a single equilibrium. Moreover, evolution even selects among the strategically stable outcomes, as Remark 2.1 reveals.

Young [25] applies the type of evolutionary process analyzed in this paper to the Nash Demand Game only. He essentially obtains the result that the tenants’ long-run share approximates \( v_* / 2 \) if the only available pie is \( v_* \). Proposition 4.1 shows that the long-run prediction of the shares can be quite different if alternative investment possibilities of the tenants are included in the analysis. In the context of subgame consistency, Proposition 4.1 shows that there exists no subgame consistent solution concept that, for each Nash Demand Game and each Investment Demand Game, selects an equilibrium whose outcome is a long-run outcome. In this sense, evolution violates subgame consistency.

The central ideas of the proof of Proposition 4.1 can be sketched as follows. Any long-run outcome is an equilibrium outcome of the Investment Demand Game because behavior is mainly best-reply driven, the Nash Demand Game is acyclic, and nobody samples more than half of the memory (cf. Young [24, 25]). Given the process is in a state with an equilibrium outcome, the memories of out-of-equilibrium subgames may “drift”\(^{10}\) in the following way: whenever a tenant experiments with a non-equilibrium investment and some high demand, the respective out-of-equilibrium memory changes according to this demand; such experiments may occur from time to time; at some point, the landlords will start to reply with a low demand each time the tenant experiments with the

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\(^{10}\) A formal definition of drift will be given in Appendix A. Roughly, a “drift” is a sequence of transitions such that in each transition at most one agent experiments, and experiments are made only in those states which the process could not leave without experiments. Because experiments are rare, “drifts” are much more likely than other transition sequences.
non-equilibrium investment and a high demand; eventually, the out-of-equilibrium memory may drift to the point where it becomes a best reply for a tenant to switch to the former non-equilibrium investment level. In this way, the process can drift away from all equilibrium outcomes except the strategically stable outcomes, because these are supported by any demands off the equilibrium path (Remark 2.1). Even the strategically stable outcomes are potentially susceptible to drift because experimental demands can occur in the $v_*$-subgame. However, the assumption of a large sample size $k$ assures that a single deviating demand does not alter the best replies, and thus it cannot initiate a drift. Hence, the process cannot drift away from the strategically stable outcomes. We complement this result with the proof that, starting from any equilibrium outcome, there exists a drift that leads to one of the strategically stable outcomes. Therefore, the long-run outcomes must be among the strategically stable outcomes.

The rest of the proof concerns mainly the Nash Demand Game following efficient investment. So we can make use of Young's [25] arguments which show that if the sample size $k$ is sufficiently large, there is an evolutionary pressure towards the tenants' shares close to $v_*/2$. This suggests that the long-run tenants' share of the efficient pie $v_*$ is approximately the closest to $v_*/2$ among all shares belonging to a strategically stable outcome. And this is what Proposition 4.1 says.

Our analysis carries over to a situation where the Nash Demand Game is played after one party has had the possibility to burn money (cf. Ben-Porath and Dekel [2]). The only change we have to make in our notation is that there exist several pies with identical values, but different costs. For instance, suppose there are two investment possibilities: one creates the pie $v_1$ at cost $C_1 = 0$, while the other creates the pie $v_2 = v_1$ at cost $C_2 > 0$ (the latter is the burning-money possibility). A variant of Proposition 4.1 implies that in the long run player 1 creates $v_1$ at no cost. Moreover, if $C_2 < \delta$ then player 1 appropriates the whole pie $v_1$ up to the base unit $\delta$.

Our model can also be extended to the case where both players have investment possibilities; we conjecture that if investments can be “fine tuned” then multiple investment profiles will prevail in the long run.\footnote{The crucial idea is that in every equilibrium outcome there is one player who gets at most half of the pie. This suggests that there exists a drift such that this player eventually expects to appropriate the entire pie after some deviating investment. In this way, the process could drift away from any investment profile.}

Finally, let us compare the long-run prediction with the prediction of equal distribution of the ex ante surplus $v_* - C(v_*)$. Such an equity standard would imply that the tenants get the share

$$x = \frac{v_* + C(v_*)}{2}$$

The crucial idea is that in every equilibrium outcome there is one player who gets at most half of the pie. This suggests that there exists a drift such that this player eventually expects to appropriate the entire pie after some deviating investment. In this way, the process could drift away from any investment profile.
of $v_e$. Both $x$ and the long-run prediction $\max\{x^*, v^*/2\}$ are (weakly) increasing in $C(v_e)$. In this sense, both confirm Rabin's [14] description of equity theory which we quoted in the Introduction.

Notice, however, that the equity standard $x$ violates the principle of opportunity costs: $x$ changes if a constant is added to the cost function $C(\cdot)$. The long-run prediction is independent of such a constant. Therefore, we cannot expect that the long-run prediction is meaningfully related to the equity standard. Indeed, $\max\{x^*, v^*/2\}$ can be larger or smaller than $x$.

5. RELATED LITERATURE

Nölkeke and Samuelson [12] study, like we, an evolutionary process for extensive form games which is based on randomly perturbed best replies to finite memories, and updating memories according to the observed path of play. However, the perturbations they assume are random mutations of memories, while we assume random experimental actions. Nölkeke and Samuelson study backward and forward induction properties of long-run outcomes. They do not consider the Investment Demand Game, but—using similar techniques as the present paper—one can show that Nölkeke and Samuelson’s model yields the same long-run outcomes as Proposition 4.1, provided the population sizes, $|A|$ and $|B|$, are sufficiently large, and $|A| = |B|$. For other games the predictions can be different. For instance, it is straightforward to extend our model to the three-stage game Nölkeke and Samuelson describe in [12, Example 2]. For that game one can show that our model (but, as they show, not their’s) yields the backward induction outcome as the unique long-run outcome.

Several authors have investigated whether evolutionary dynamics support forward induction and strategic stability in particular. Nölkeke and Samuelson [12, Section 5] study a class of outside option games which includes the Nash Demand Game with an (unilateral) outside option. For the example of symmetric $2 \times 2$ games with an outside option, they illustrate the techniques by which the long-run outcomes can be determined. For the Nash Demand Game with pie $v$ and a player-1 outside option with value $o \in (0, v)$, such techniques (combined with those of Young [25]) imply that the long-run outcome is (up to the base unit $\delta$) the deal-me-out outcome: the pie $v$ is provided, and player 1’s share

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12 This is the main result of Ellingsen and Robles [5]. My thesis (Tröger [22]) contains a very similar result, using a variant of Nölkeke and Samuelson’s model. [5] also analyze the ultimatum game with prior investment.
approximates \( \max\{o, v/2\} \). The same holds in our model. Swinkels [20] considers myopic adjustment dynamics, a generalization of the replicator dynamic. He provides conditions for asymptotically stable sets to contain strategically stable and hyperstable sets. Nöldeke and Samuelson [13] undertake an evolutionary analysis of signaling games, using a variant of the model developed in [12]. They find that the long-run outcomes are closely related to several signaling refinements which are themselves closely related to strategic stability.

From Young [25] our paper borrows the technique to analyze the Nash Demand Game. Young’s analysis of this game shows that the pie is split equally in the long-run as long as all agents have the same sample size \( k \); if, however, every tenant’s sample size is much larger (smaller) than every landlord, then the tenants (landlords) get the lion share of the pie in the long-run. Reference [25] also shows that the effect of risk aversion on the long-run shares of the pie resembles the Nash Bargaining Solution.

Agastya [1] applies a variant of Young’s [25] model to double auction games. In contrast to the Nash Demand Game with pie \( v \) and demands \( x \) and \( y < v - x \), in a double auction the share \( v - x - y \) is not lost, but assigned to the two players in fixed proportions. Reference [1] shows that the long-run shares depend on these proportions; i.e., the long-run shares are sensitive to a slight variation in the specification of the bargaining game which does not change the equilibrium set.

**APPENDIX A**

In this section, \( (P^\epsilon)_{\epsilon \in (0,1)} \) denotes any regular perturbation (as defined by Young [24, p. 77]), and \( \Theta \) denotes a state space. The main result of this

13 Binmore et al. [4] present an experimental study of this game which confirms the deal-me-out outcome if \( o \) is not too large. They also provide an explanation for the failure of the deal-me-out outcome if \( o \) is large. This explanation is based on the idea that some player-2 agents may tremble around the best reply demand, which is more costly for a not-opting-out player-1 agent if \( o \) is larger.

14 To see this, suppose there are two pie sizes, \( v \) and 0, with costs \( o \) and \( -o \), respectively. Formally, this case is not captured by Proposition 4.1, because we have assumed for presentation convenience that all feasible pies have at least size \( 2k \).

15 Extending Proposition 4.1 to the case of heterogeneous sample sizes is straightforward if the smallest sample size in the tenants’ population is not larger than the smallest sample size in the landlords’ population. Surprisingly, new technical problems arise in the case where a landlord has the smallest sample among all agents. The reason is that, starting from a strategically stable outcome, the least number of experiments such that an agent’s best reply changes, might be a row of demands \( \delta \) by tenants in the \( v_\epsilon \)-subgame; given this row is sufficiently long to change a landlord’s best reply demand to \( v_\epsilon - \delta \), it is then still not clear whether the process can settle on a new strategically stable outcome without further experiments; even if enough experiments occur for a new strategically stable outcome to be reached, it is not clear which one.
section is Proposition A.1 which characterizes the long-run states of \((P^\epsilon)_{\epsilon \in (0,1)}\). Before we can formulate the result, we have to make some definitions. To improve the readability of the definitions, we indent those paragraphs which apply graph theory to regular perturbations.

Consider any finite set \(Z \neq \emptyset\). For each pair \((z, z') \in Z^2\), let a number \(d(z, z') \in [0, 1, \ldots, \infty]\) be given.

For any \(z_1, \ldots, z_n \in Z (n \geq 2)\), we call \(p = (z_1, \ldots, z_n)\) a path from \(z_1\) to \(z_n\), and call \(d(p) := \sum_{i=1}^{n-1} d(z_i, z_{i+1})\) its resistance. We define \(d(z, z') = \min_{p \text{ is a path from } z \text{ to } z'} d(p)\).

We call \(z\) and \(z'\) \(d\)-equivalent if there exists a path of 0 resistance from \(z\) to \(z'\) and vice versa, i.e., if \(\tilde{d}(z, z') = 0\) and \(\tilde{d}(z', z) = 0\). A \(d\)-equivalence class \(W \subseteq Z\) is called stable if \(\forall z \in W, z' \in Z \setminus W : \tilde{d}(z, z') > 0\).

The set of stable \(d\)-equivalence classes is nonempty, and is denoted \(\mathcal{W}(Z, d)\).

Let \(r(d, s, s')\) be the minimum number of experiments over all transitions from \(s \in \Theta\) to \(s' \in \Theta\), i.e., \(P^s_{s'} = \emptyset (s, s') \) if \(P^s_{s'} > 0\), and \(r(d, s, s') = \infty\) if \(P^s_{s'} = 0\). The elements of \(R_0 = \mathcal{W}(\Theta, r_0)\) are called 0-recurrent sets.\(^{16}\)

For all \(W, W' \in \mathcal{W}(Z, d)\) with \(W \neq W'\), we define \(d'(W, W) = 0\) and \(d'(W, W') = \tilde{d}(z, z') - 1\), with arbitrary \(z \in W\) and \(z' \in W'\) (the definition is independent of \(z\) and \(z'\)). Notice that \(d'(W, W') > 0\).

For all \(i = 1, 2, \ldots\), let \(r_i := r_{i-1}\) and \(R_i := \mathcal{W}(R_{i-1}, r_i)\). The elements of \(R_i\) are called \(i\)-recurrent sets.\(^{17}\) We introduce a notational convention for all \(i \geq 0\): if \(X\) is any set whose elements are \(i\)-recurrent sets, then \(X\) will be identified with the union of its elements, i.e., \(X = \bigcup_{x \in X} x\). Repeated application of this convention identifies each \(i\)-recurrent set with a subset of \(\Theta\).

For any \(C, C' \in R_0\) with \(r_1(C, C') = 0\), we say that there exists a drift from \(C\) to \(C'\).

Let \(G \subseteq Z^2\) be a directed graph. The resistance of \(G\) is defined as

\[
v(G, d) = \sum_{(z', z'') \in G} d(z', z'').
\]

\(^{16}\) These are the absorbing sets of the process \(P^\epsilon\). Young [24] calls these sets recurrent communication classes.

\(^{17}\) Nöldeke and Samuelson’s [12] definition of a locally stable component is equivalent to that of a 1-recurrent set.
Let \( p = (z_1, \ldots, z_n) \) be a path. We call \( p \) a path in \( G \) if
\[
\{(z_1, z_2), \ldots, (z_{n-1}, z_n)\} \subseteq G.
\]
Let \( z \in Z \). We call \( G \) a \( z \)-graph if for all \( z' \in Z \setminus \{z\} \), there exists a path from \( z' \) to \( z \) in \( G \). A \( z \)-graph is called a \( z \)-tree if it is minimal with respect to \( \subseteq \). Notice that each \( z \)-tree has exactly \(|Z| - 1\) edges. The stochastic potential of \( z \) is defined as
\[
v(z, d) = \min_{T \text{ is a } z \text{-tree}} v(T, d) = \min_{G \text{ is a } z \text{-graph}} v(G, d).
\]
The elements of \( Z \) with minimum stochastic potential are denoted
\[
\mathcal{M}(Z, d) = \arg \min_{z \in Z} v(z, d).
\]
The main result, promised above, is that for all \( i \geq 0 \), the set of long-run states is the union of the \( i \)-recurrent sets with minimum stochastic potential.

**Proposition A.1.** Let \( (P^n)_{n \in \{0, 1\}} \) be a regular perturbation, and let \( i \geq 0 \). Then the set of long-run states is \( \mathcal{M}(R_i, r_{i+1}) \).

Note that this generalizes Young [24, Theorem 4(ii)] who shows that the set of long-run states is \( \mathcal{M}(R_0, r_1) \). Proposition A.1 also generalizes Nöldeke and Samuelson [12, Proposition 1] who show that the set of long-run states is the union of some of the \( 1 \)-recurrent sets.

**Proof of Proposition A.1.** By Lemma A.1 and Lemma A.2 below, the set of long-run states is
\[
\mathcal{M}(\Theta, r_0) = \mathcal{M}(R_0, r_1) = \mathcal{M}(R_1, r_2) = \cdots = \mathcal{M}(R_i, r_{i+1}).
\]

**Lemma A.1 (Young [24, Lemma 1] and Samuelson [16, Theorem 1 (1.2)]).** Let \( (P^n)_{n \in \{0, 1\}} \) be a regular perturbation. Then the set of long-run states is \( \mathcal{M}(\Theta, r_0) \), where \( \Theta \) denotes the state space.

**Lemma A.2 (Based on Young [24, Lemma 2]).** Let \( Z \neq \emptyset \) be a finite set. For each pair \((z, z') \in Z^2\), let a number \( d(z, z') \in \{0, 1, \ldots, \infty\} \) be given. Then, we have
\[
\mathcal{M}(Z, d) = \mathcal{M}(\mathcal{H}(Z, d), d'),
\]
where the set on the r.h.s. is identified with the union of its elements.
Proof of Lemma A.2. For each $W \in \mathcal{W}(Z, d)$, fix a $z_W \in W$ and call it the special vertex in $W$. We show first that

$$\mathcal{A}(Z, d) \cong \mathcal{W}(Z, d),$$

where the r.h.s. is identified with the union of its elements.

To prove this, let $z \in Z \setminus \mathcal{W}(Z, d)$. There exists a $z$-graph $G$ with $v(G, d) = v(z, d)$, and a path $(z_1, \ldots, z_n)$ $(n \geq 2)$ with resistance 0 from $z$ to $z_W$ for some $W \in \mathcal{W}(Z, d)$. Let $(z'_1, \ldots, z'_n)$ be a path from $z_W$ to $z$ in $G$, and let $i$ be minimal with $z'_i \notin W$. Because $d(z'_{i-1}, z'_i) > 0$ the set

$$G' = (G \cup \{(z_1, z_2), \ldots, (z_{n-1}, z_n)\}) \setminus \{(z'_{i-1}, z'_i)\}$$

is a $z'_{i-1}$-graph with $v(G', d) < v(G, d)$. Hence, $v(z'_{i-1}, d) < v(z, d)$, and thus $z \notin \mathcal{A}(Z, d)$.

A similar argument shows that $v(z, d) = v(z', d)$ for all $W \in \mathcal{W}(Z, d)$ and $z, z' \in W$. Hence, it remains to show (**).

(**) Let $W \in \mathcal{W}(Z, d)$. Then

$$v(z_W, d) = v(W, d') + |\mathcal{W}(Z, d)| - 1.$$

We prove “=” in (**) by showing “≤” and “≥.”

“≤” Let $\tau$ be a $W$-tree with $v(\tau, d') = v(W, d')$. For every $\eta =: (W', W'') \in \tau$, there exists a path $(z_{\eta 1}, \ldots, z_{\eta n})$ $(n \geq 2)$ from $z_{W'}$ to $z_{W''}$ with resistance

$$d'(\eta) + 1 = \sum_{i=1}^{n-1} d(z_{\eta i}, z_{\eta i+1}).$$

Moreover, for every $e \in Z$, the definition of $\mathcal{W}(Z, d)$ implies that there exists $W^e \in \mathcal{W}(Z, d)$, and a path $(z_{\eta 1}, \ldots, z_{\eta n})$ $(n \geq 2)$ from $e$ to $z_{W^e}$ with 0 resistance. The set

$$G = \{(z_{\eta 1}, z_{\eta 1+1}) \mid \eta \in \tau, i < n\} \cup \{(z_{\eta i}, z_{\eta i+1}) \mid e \in Z \setminus \{z_{W^e}\}, i < n\}$$

is a $z$-$\mathcal{W}$-graph. We have

$$v(G, d) \leq \sum_{\eta \in \tau} (d'(\eta) + 1) = v(z, d') + |\mathcal{W}(Z, d)| - 1.$$

Thus,

$$v(z_W, d) \leq v(G, d) \leq v(W, d') + |\mathcal{W}(Z, d)| - 1.$$
“≥” We start with three definitions. Given a tree $T$, we call $e \in Z$ a junction in $T$ if there exist $z', z'' \in Z$ such that $(z', e) \in T$ and $(z'', e) \in T$. The number of junctions that are not special vertices is denoted $N_T$. For any $z, z' \in Z$, we call $z$ a predecessor of $z'$ in $T$ if $z = z'$ or there exists a path from $z$ to $z'$ in $T$.

Let $T$ be a $z_w$-tree such that $N_T$ is minimal under the condition

\[ v(T, \bar{d}) \leq v(z_w, d). \]

(A.1)

We will first show that $N_T = 0$. (A.2)

Suppose that $N_T > 0$. Let $e$ be a junction in $T$ that is not a special vertex. There exists a $W' \in \mathcal{W}(Z, d)$ such that $\bar{d}(e, z_w) = 0$. We define the $z_w$-tree

\[ T' = (T \cup \{(z, z_w') \mid z \in J\}) \setminus \{(z, e) \mid z \in J\}, \]

where

\[ J = \{ z \in Z \mid (z, e) \in T, z_w' \text{ is not a predecessor of } z \text{ in } T \}. \]

For all $z \in J$, we have

\[ \bar{d}(z, z_w') = \bar{d}(z, e) + \bar{d}(e, z_w') = \bar{d}(z, e), \]

and therefore $v(T', \bar{d}) \leq v(T, \bar{d}) \leq v(z_w, d)$ by (A.1). Moreover, $e$ is not a junction in $T'$, implying $N_T = N_T - 1$, which yields a contradiction.

Now we define a $W$-tree $\tau$. For any $W', W'' \in \mathcal{W}(Z, d)$, let $\eta = (W', W'') \in \tau$ if and only if there exists a path $p^\eta = (z_1', ..., z_n')$ $(n' \geq 2)$ from $z_w'$ to $z_w$ in $T$ such that $z_2', ..., z_{n'-1}'$ are not special vertices. By construction, $\tau$ is a $W$-tree. By (A.2), the paths $p^\eta$ $(\eta \in \tau)$ are pairwise edge-disjoint. Hence,

\[ v(W, d') \leq v(\tau, d') \leq \sum_{\eta \in \tau} (d(p^\eta) - 1) \leq v(T, \bar{d}) - (|\mathcal{W}(Z, d)| - 1). \]

Together with (A.1) this implies “≥” in (**).}

**APPENDIX B**

For any $n \geq 1$, $x$ and $y$, we use the shortcut

\[ [x, y]_n = ((x, y), ..., (x, y)). \]

\[ \text{\textit{n times}} \]
For any feasible outcome \((v, x, v-x)\), we denote by \(Q(v, x)\) the set of states \(s \in \Theta\) with the following properties: \((v, x)\) is the unique best reply of the tenants to \(s\), and the \(v\)-memory is coordinated on the equilibrium \((x, v-x)\) (i.e., \(s(v) = [x, v-x]_m\)).

We define
\[
X = (0, v_\ast), \quad X(\delta) = X \cap \{\delta, 2\delta, \ldots\}, \quad X_\ast(\delta) = X(\delta) \cap \{x \mid x > x_\ast - \delta\}.
\]

For each \(x \in X_\ast(\delta)\), we define the state \(s^*_x\) by \(s^*_x(v) = [x, v-x]_m\), and \(s^*_x(v) = [v-\delta, \delta]_m\) for all \(v \neq v_\ast\). For all \(x \in X\), we define
\[
\bar{m}_1(x) = \left[ k \frac{x}{v_\ast - \delta} \right], \quad \bar{m}_1(x) = \left[ k \frac{\delta}{x} \right],
\]
\[
\bar{m}_2(x) = \left[ k \frac{\delta}{v_\ast - x} \right], \quad \bar{m}_2(x) = \left[ k \frac{v_\ast - x}{v_\ast - \delta} \right].
\]

(Here, \(\lfloor \cdot \rfloor\) denotes the smallest integer larger or equal to \(\cdot\)). It is straightforward that
\[
\bar{m}_2 \left( \frac{v_\ast}{2} \right) = \bar{m}_1 \left( \frac{v_\ast}{2} \right), \quad \bar{m}_2 \left( \frac{x}{2} \right) = \bar{m}_1 \left( \frac{x}{2} \right).
\]

Moreover, given \(\delta\), the following holds for all sufficiently large \(k\):
\[
k \geq \frac{2(v_\ast - \delta)}{\delta},
\]
\[
\forall x \in X(\delta) : \bar{m}_2(x) \leq \bar{m}_1(x), \quad \forall x \in X(\delta) : \bar{m}_1(x) \leq \bar{m}_2(x),
\]
\[
\bar{m}_2(x) \text{ is weakly increasing on } X, \quad \bar{m}_1(x) \text{ is weakly decreasing on } X.
\]

The first lemma below says that each subgame-memory can be coordinated without any experiments on some equilibrium. The proof is omitted because it follows the same steps as the proof of \([25, \text{Theorem 1}]\).
Lemma B.1. Let \( v' \in \mathcal{V} \), and let \( s'(v') \) be a \( v' \)-memory. Then, there exists a sequence \( s^1(v'), \ldots, s^{n(v')}(v') \) (with \( n(v') \geq 1 \)) of pairwise different \( v' \)-memories such that (i) and (ii) hold.

(i) There exists a demand \( x(v') \) such that \( s^{n(v')}(v') = [x(v'), v'-x(v')]_m \).

(ii) For all \( j = 1, \ldots, n(v') - 1 \), if \( s^j(v') \) is the prevailing \( v' \)-memory at the beginning of some period \( t' \), then with positive probability the prevailing \( v' \)-memory at the beginning of period \( t'+1 \) is \( s^{j+1}(v') \), conditional on the event that the \( v' \)-subgame is entered in period \( t' \) and best reply demands are made.

The next lemma characterizes the 0-recurrent sets.

Lemma B.2. We have \( C \in R_0 \) if and only if there exists a feasible outcome \( (v, x, v-x) \) such that \( C = \{s\} \) for some \( s \in Q(v, x) \).

Proof. (If) Suppose that at the beginning of some period the process is in state \( s \in Q(v, x) \). Unless an experiment occurs, the outcome will be \( (v, x, v-x) \), and the process will remain in state \( s \).

(Only If) We show that for each \( s \in \Theta \) there exists a sequence \( s_1, \ldots, s_n \in \Theta \) (\( n \geq 1 \)) with \( s_1 = s \), \( r_0(s_i, s_{i+1}) = 0 \) (\( i = 1, \ldots, n-1 \)), and \( \{ s_n \} \in Q(v, x) \) for some feasible outcome \( (v, x, v-x) \). (In connection with the result of “IF,” this implies “ONLY IF.”)

Fix any \( s \in \Theta \). For each \( v' \in \mathcal{V} \), apply Lemma B.1 to complete \( s(v') =: s^1(v') \) to a sequence with the properties (i) and (ii). We now define an infinite sequence of states \( s_1, s_2, \ldots \) with \( r_0(s_i, s_{i+1}) = 0 \) (\( i \geq 1 \)) by running in an appropriately alternating order through the sequences selected in Lemma B.1, and repeating \( s^{n(v')}(v') \) when reached. Formally, let \( s_1 = s \); for all \( i \geq 1 \), fix a best-reply pie \( w_i \in \mathcal{V} \) in state \( s_i \), and let \( s_{i+1} \) for all \( v' \in \mathcal{V} \) be defined by

\[
s_{i+1}(v') = \begin{cases} 
  s_i(v'), & \text{if } v' \neq w_i \\
  s^{i+1}(v'), & \text{if } v' = w_i \text{ and } s_i(v') = s^i(v') \text{ and } j < n(v'), \\
  s^{n(v')}(v'), & \text{if } v' = w_j \text{ and } s_i(v') = s^j(v') \text{ and } j = n(v')
\end{cases}.
\]

Lemma B.1 implies \( r_0(s_i, s_{i+1}) = 0 \) for all \( i \). Moreover, there exist \( i_1 \) and \( u_1 \in \mathcal{V} \) such that \( s_{i_1}(u_1) = s^{n(u_1)}(u_1) \), and \( u_1 \) is a best-reply pie in state \( s_{i_1} \). Now, if \( u_1 \) is the unique best-reply pie in \( s_{i_1} \), then we define \( n = i_1, v = u_1 \) and \( x = x(u_1) \), and are ready with the proof in this case. Otherwise, for all \( i > i_1 \), we assume without loss of generality that \( w_i \neq u_i \) whenever \( u_i \) is not
the unique best-reply pie in state $s_i$. Therefore, there exist $t_2 > i_1$ and $u_2 \in \mathcal{V}$ such that $s_{u_2}(u_2) = x^m(u_2)$ and either

$$u_2 = u_1 \quad \text{and} \quad u_1 \text{ is the unique best-reply pie in state } s_{u_1},$$

or

$$u_2 \neq u_1 \quad \text{and} \quad u_2 \text{ is a best-reply pie in } s_{u_1}, \quad \text{but} \quad u_1 \text{ is not.}$$

(Here we have used our assumption from p. 5 that indifferences are excluded for player 1.) In the “either” case, we define $n = t_2$, $v = u_1$ and $x = x(u_1)$, and are ready with the proof in this case. In the “or” case, if $u_2$ is the unique best-reply pie, we are again ready, defining $n = t_2$, $v = u_2$ and $x = x(u_2)$. Otherwise, for all $i > t_2$, we assume without loss of generality that $w_i \notin \{u_1, u_2\}$ whenever neither $u_1$ nor $u_2$ is the unique best-reply pie in state $s_i$ (note that on p. 5 we have excluded that player 1 is indifferent between $u_1$ and $u_2$). Using induction, it is clear that this reasoning can be continued. At the latest, it ends with some $n = t_{i(v')}$, and $s_n = (x^m(v'))_{v' \in \mathcal{V}}$. In state $s_n$, a unique best-reply pie $u_n$ exists because player 1 is not indifferent between any two pies by our assumption on p. 5; hence, we can define $v = u_n$ and $x = x(u_n)$ to complete the proof in this case.

The next lemma says that by an appropriate drift any out-of-equilibrium memory can be coordinated on an equilibrium in the respective subgame. During the coordination, the optimal investment level may change.

**Lemma B.3.** Let $(v, x, v-x)$ be a feasible outcome, $s \in Q(v, x)$, and $v' \in \mathcal{V} \setminus \{v\}$. Then, there exists a state $s' \in R_0$ and $x'$ such that $\bar{f}_i(s, s') = 0$, $s'(\bar{v}) = s(\bar{v})$ for all $\bar{v} \neq v'$, and $s'(v') = [x', v' - x']_m$.

In particular, $s' \in Q(v, x) \cup Q(v', x')$.

**Proof.** First apply Lemma B.1 to complete $s(v') =: s'(v')$ to a sequence with the properties (i) and (ii). For $i = 1, ..., n(v')$, define $s_i$ by $s_i(v') = s'(v')$, and $s_i(v) = s(v)$ for all $\bar{v} \neq v'$. For all $i$ with $s_i \notin R_0$ we have $r_d(s_i, s_{i+1}) = 0$ (because $v'$ is a best reply pie in state $s_i$), whereas for all $i$ with $s_i \in R_0$ we have $r_d(s_i, s_{i+1}) = 1$ (because the tenant may experiment with the pie $v'$ followed by a best reply demand). Hence, $\bar{f}_i(s, s_n(v')) = 0$, and the proof is completed by the definitions $s' = s_n(v')$ and $x' = x(v')$.

The next lemma assumes that one out-of-equilibrium memory is already coordinated on an equilibrium in that subgame. It is shown that re-coordination on any other equilibrium in that subgame is possible by an appropriate drift. During the re-coordination, the optimal investment level may change.
**Lemma B.4.** Let \((v, x, v-x)\), \((v', x', v'-x')\), and \((v', y, v'-y)\) be feasible outcomes with \(v' \neq v\). Let \(s' \in Q(v, x)\) with \(s'(v') = [x', v'-x']_m\), and let \(s_i\) be defined by \(s_i(v') = [y, v'-y]_m\), and \(s_i(e) = s'(e)\) for all \(e \neq v'\). Then, \(r_i(s', s_i) = 0\).

In particular, \(s_0 \in Q(v, x) \cup Q(v', y)\).

**Proof.** As in the previous lemmata, we will define a certain sequence of states \(s_1, ..., s_n\) (\(n\) will be determined below). For all \(i\) and \(e \neq v'\), let \(s_i(e) = s'(e)\). Let (read the r.h.s. as a concatenation)

\[ s_i(v') = [y, v'-y]_{i-1} [x', v'-x']_{m-i+1} \]

for \(i \leq i^*\), where \(i^*\) is minimal with the property that \(v' - y\) is a best reply demand of the landlords in the \(v'\)-subgame if the prevailing \(v'\)-memory is \(s_i(v')\). For \(i = i^* + 1, ..., n\), where \(n = i^* + m\), let \(s_i(v')\) be the first \(m\) demand pairs of

\[ [y, v'-y]_{i-1} [x', v'-x']_{m-i+1} \]

For all \(i\), if \(s_i \notin R_0\) then \(v'\) is a best reply in \(s_i\), and \((v', x')\) gives the tenants less payoff than \((v, x)\), hence \((v', y)\) is a best reply, implying \(r_0(s_i, s_{i+1}) = 0\). On the other hand, for all \(i\) with \(s_i \in R_0\) we have \(r_0(s_i, s_{i+1}) = 1\) (the tenant may experiment with \((v', y)\)). Therefore, we have \(r_i(s', s_n) = 0\). The observation \(s_n = s_\alpha\) completes the proof.

The next lemma establishes that the 1-recurrent sets correspond to the strategically stable outcomes.

**Lemma B.5.** For each demand \(x \in X_{\geq}(\delta)\), there exists a unique 1-recurrent set, \(C(x)\), such that \(s^*_x \in C(x) \subseteq Q(v_x, x)\). We have

\[ R_1 = \{ C(x) \mid x \in X_{\geq}(\delta) \}. \]

**Proof.** **Step 1** below shows that, for each \(x \in X_{\geq}(\delta)\), no drift from a state in \(Q(v_x, x)\) to a state outside \(Q(v_x, x)\) exists. Hence, a 1-recurrent set within \(Q(v_x, x)\) exists. **Step 2** shows that from any state in \(Q(v_x, x)\) a drift to \(s^*_x\) exists. Hence, each 1-recurrent set within \(Q(v_x, x)\) contains \(s^*_x\), and there is a unique such set. We call it \(C(x)\). **Step 3** shows that from any 0-recurrent state a drift into some \(Q(v_x, x)\) \((x \in X_{\geq}(\delta))\) exists. Hence, there exist no further 1-recurrent sets beyond the sets \(C(x)\) \((x \in X_{\geq}(\delta))\).

**Step 1.** For all \(x \in X_{\geq}(\delta)\), \(s \in Q(v_x, x)\), and \(s' \in R_0\) such that \(r_1(s, s') = 0\), we have \(s' \in Q(v_x, x)\).

To see this, suppose the process is in some state \(s \in Q(v_x, x)\) with \(x \in X_{\geq}(\delta)\), and an experiment \((v', x')\) of a tenant or an experiment \(y'\) of
a landlord occurs. In the case of a tenant’s experiment with \( v'' = v_\star \), or a landlord’s experiment, inequality (B.6) implies that after the experiment, \((x, x_\star - x)\) are still the unique best reply demands in the \( x_\star \)-subgame, and, due to (B.7), \( v_\star \) is still the unique best reply pie. Hence, \( s' = s \) for all \( s' \in R_0 \) with \( r_1(s, s') = 0 \). In the remaining case of a tenant’s experiment with \( v'' \neq v_\star \), the \( v''\)-memory might change, but \( v_\star \) remains the unique best reply pie, because \( x \in X_\star (\delta) \). Hence, while \((x'') \neq s''(v'')\) is possible for some \( s'' \in R_0 \) with \( r_1(s, s'') = 0 \), we still have \( s'' \in Q(v_\star, x) \).

***Step 2.*** Let \( x \in X_\star (\delta) \) and \( s \in Q(v_\star, x) \). Then \( r_1(s, s_\star') = 0 \).

This follows from Lemmata B.3 and B.4. More precisely, fix any \( v' \neq v_\star \) and apply Lemma B.3 with \( v := v_\star \). We have \( s' \in Q(v_\star, x) \) because \( x \in X_\star (\delta) \). Hence, we can apply Lemma B.4 with \( v := v_\star \) and \( y := v' - \delta \). This yields \( r_1(s', s'') = 0 \), where we define \( x' := x_\star \). Altogether, \( r_1(s, s') = r_1(s', s'') = 0 \), and \( s' = s_\star (\delta) \) for \( \delta \in \{ v_\star, v' \} \). Repeating these arguments for all \( v' \neq v_\star \) yields \( r_1(s, s_\star') = 0 \).

***Step 3.*** Let \( s \in R_0 \). Then, there exists \( \hat{x} \) and \( \delta \in \check{Q}(v_\star, \hat{x}) \) such that \( r_1(s, \hat{x}) = 0 \).

Let \((x, y)\) be such that \( s \in Q(v, x) \). We define \( \hat{x} \) by

\[
\{ \hat{x} \} = \arg \max \{ v' \in x \setminus \{ v \} \} \ (v' - C(v')).
\]

We start with showing (*):

(*) There exist \( x' \in R_0 \), \( x_\star' \), and \( \hat{x} \), such that \( r_1(s, s') = 0 \), \( s'(v_\star') = [x_\star, x_\star - x_\star] \), and \( s'(\hat{x}) = [\hat{x}, \hat{x} - \hat{x}] \).

To see (*), distinguish three cases. First, if \( v = v_\star \), set \( x_\star = x \), apply Lemma B.3 with \( v := v_\star \) and \( y := v - \delta \). This yields \( s_\star = s_\star \), and the definitions \( \hat{x} = v_\star - \delta \). Second, if \( v = v_\star \) then set \( \hat{x} = x \), apply Lemma B.3 with \( v := v_\star \) and \( y := v_\star - \delta \). This yields \( s_\star = s_\star \), and the definitions \( \hat{x} = v_\star - \delta \). The third case, where \( v \notin \{ v_\star, \hat{x} \} \), is similar, but Lemma B.3 must be applied twice in succession.

Now, given (*), either \( s' \notin Q(v_\star, x_\star) \) or \( s' \notin Q(\hat{x}, x) \). In the “Either-case” apply Lemma B.4 with \( v \) and \( x \) defined by \( s' \in Q(v, x) \), \( x' := x_\star \), and \( y := v_\star - \delta \). This yields \( r_1(s, s') = 0 \), and the definitions \( \hat{x} = v_\star - \delta \), \( \hat{x} = s_\star - \delta \) complete this case. In the “or”-case apply Lemma B.4 with \( v \) and \( x \) defined by \( s' \in Q(v, x) \), \( x' := \hat{x} \), and \( y := v_\star - \delta \). This yields \( r_1(s, s') = 0 \), where we define \( \hat{x} = x_\star - \delta \). Now, if \( x' \in Q(v_\star, x_\star) \), then \( x_\star = x_\star \), and the definitions \( \hat{x} = x_\star - \delta \), \( \hat{x} = s_\star \) complete this case. Otherwise, we have \( x' \in Q(\hat{x}, \hat{x} - \delta) \). Then, apply Lemma B.4 with \( s' := s' \), \( v = v_\star \), \( x = \hat{x} \), \( x' := x_\star \), and \( y := v_\star - \delta \). The definitions \( \hat{x} = v_\star - \delta \), \( \hat{x} = s_\star - \delta \) complete the proof.
Define $\hat{x}$ as the largest $x \in X(\delta)$ with $x \leq v_*/2$. We define $\hat{y} = \hat{x}$ if $\hat{x} \in X_\rightarrow(\delta)$; otherwise we define $\hat{y}$ as the smallest $x \in X_\rightarrow(\delta)$.

**Lemma B.6.** We have $|x - \max\{x_*, v_*/2\}| \leq \delta$ for $x = \hat{y}$ and (if $\hat{y} \neq v_*/2$) for $x = \hat{y} + \delta$.

**Proof.** Consider first the case $x_* \leq v_*/2$. Then, $x_* - \delta < \hat{y} \leq v_*/2$, which proves the claim. Second, consider the case $x_* - \delta < v_*/2 < x_*$. Because $(x_* - \delta, x_*)$ contains an integer multiple of $\delta$ we have $\hat{y} \in (x_* - \delta, x_*)$, which proves the claim. Finally, consider the case $x_* - \delta \geq v_*/2$. Again, $\hat{y} \in (x_* - \delta, x_*)$.

The basic idea of the following lemma is from Young [25, p. 161]. The notation refers to that introduced in Appendix A. The proof is straightforward and hence omitted.

**Lemma B.7.** For any $z \in Z$, let
\[
\begin{align*}
\alpha'(z) & = \arg \min_{z' \in Z(z)} d(z, z'), \\
\eta(z) & = \min_{z' \in Z(z)} d(z, z'), \\
\alpha' & = \arg \max_{z \in Z} \eta(z).
\end{align*}
\]

Let $\mathcal{S}$ be the set of $z \in Z$ with the following property: there exists a $z$-tree $T$ such that $z' \in \alpha'(z')$ for all $(z', z'') \in T$.

If $\mathcal{N} \cap \mathcal{S} \neq \emptyset$ then $\delta(Z, d) = \alpha' \cap \mathcal{S}$.

The final two lemmata concern the calculation of $\mathcal{N} \cap \mathcal{S}$ in the case $Z := R_1$ and $d := r_2$. We use the shortcuts $\eta(x) = \eta(C(x))$, $\mathcal{N}(x) = \mathcal{N}(C(x))$, and $\alpha' = \arg \max_{x \in X_\rightarrow(\delta)} \eta(C(x))$. In the next lemma we derive properties of $\eta(\cdot)$.

**Lemma B.8.** For all $x \in X_\rightarrow(\delta)$, we have
\[\eta(x) \geq \min\{\bar{m}_2(x), m_1(x)\} - 2,\] (B.10)
and the following entailments hold:
\[x \neq v_*/\delta \Rightarrow \eta(x) \leq \bar{m}_2(x) - 2,\] (B.11)
\[x \leq v_*/2, x \neq v_*/\delta \Rightarrow \eta(x) = \bar{m}_2(x) - 2, C(x + \delta) \in \mathcal{N}(x),\] (B.12)
\[x - \delta \in X_\rightarrow(\delta) \Rightarrow \eta(x) \leq m_1(x) - 2,\] (B.13)
\[x \geq v_*/2, x - \delta \in X_\rightarrow(\delta) \Rightarrow \eta(x) = m_1(x) - 2, C(x - \delta) \in \mathcal{N}(x).\] (B.14)
Proof. At first we define auxiliary functions $m_i(x, y)$ and $m_2(x, y)$, for all $x, y \in X(\delta)$ with $y \neq x$. Consider the set of $v_\ast$-memories in which every landlord-demand equals either $v_\ast - x$ or $v_\ast - y$, and let $m_i(x, y)$ be the minimal number of $(v_\ast - y)$-demands such that $y$ is a best reply demand of the tenants in the $v_\ast$-subgame. Similarly, consider the set of $v_\ast$-memories in which each tenant-demand equals $x$ or $y$, and let $m_2(x, y)$ be the minimal number of $y$-demands such that $v_\ast - y$ is a best reply demand of the landlords in the $v_\ast$-subgame. Straightforward calculations yield:

$$y \geq x + \delta \Rightarrow m_1(x, y) = \left( k \frac{x}{y} \right) ^\delta = m_1(x, v_\ast - \delta), \quad (B.15)$$

$$y \leq x - \delta \Rightarrow m_1(x, y) = \left[ k \left( \frac{1 - y}{x} \right) \right] ^\delta = m_1(x, x - \delta), \quad (B.16)$$

$$y \geq x + \delta \Rightarrow m_2(x, y) = \left( k \frac{y - x}{v_\ast - x} \right) ^\delta = m_2(x, x + \delta), \quad (B.17)$$

$$y \leq x - \delta \Rightarrow m_2(x, y) = \left[ k \left( \frac{v_\ast - x}{v_\ast - y} \right) \right] ^\delta = m_2(x, x - \delta). \quad (B.18)$$

Proof of (B.10). Let $x \in X_\ast(\delta)$. There exists $C \in R_1 \setminus \{C(x)\}$ such that $r_3(C(x), C) = \eta(x)$. Hence, there exists $q = (s_1, ..., s_n)$ (n $\geq 2$) such that $s_i \in R_0$ ($i = 1, ..., n$), $s_1 \in C(x)$, $s_n \in C$, and $\eta(x) = r_3(q) - 1$. There exists an $i \in \{1, ..., n - 1\}$ such that $s_i \notin \bigcap_i (C(v_\ast, x))$ and $s_{i+1} \notin \bigcap_i (C(v_\ast, x))$ (if $i$ did not exist, we would have $s_n \notin \bigcap_i (C(v_\ast, x)) \cap C = \varnothing$). There exists $p_i = (z_1, ..., z_l)$ ($l \geq 2$) such that $z_j \in \Theta$ ($j = 1, ..., l$), $z_1 = s_1$, $z_l = s_{i+1}$, and $r_3(p_i) - 1 = r_3(s_i, s_{i+1})$. Therefore, we have

$$\eta(x) \geq r_3(p_i) - 2.$$

Now consider the following conditions (a), (b), and (c), each of which may or may not be true for any state $s \in \Theta$, given $x$.

(a) There exists $y \in X(\delta) \setminus \{x\}$ such that $(v_\ast, y)$ is a best reply of the tenants to state $s$.

(b) There exists $y \in X(\delta) \setminus \{x\}$ such that $v_\ast - y$ is a best reply demand of the landlords to state $s$ in the $v_\ast$-subgame.

(c) $v_\ast$ is not a best reply pie in state $s$.

None of these conditions is fulfilled in state $z_1 = s_1$, but at least one is fulfilled in state $z_2 = s_{i+1}$ (otherwise in state $z_i$ the outcome $(v_\ast, x, v_\ast - x)$ could occur without an experiment). Let $j$ be minimal with the property that $z_j$ fulfills one of the conditions (a) to (c). In the $v_\ast$-memory $z_j(v_\ast)$,
each landlord-demand $\neq v_a - x$, as well as each tenant-demand $\neq x$, corresponds to an experiment along the path $p_i$. Hence, if (a) holds for $z_j$ then we have

$$\eta(x) \geq r_0(p_i) - 2 \geq m_1(x, y) - 2.$$  \hspace{1cm} \text{(B.19)}

Similarly, if (b) holds for $z_j$ then we have

$$\eta(x) \geq r_0(p_i) - 2 \geq m_2(x, y) - 2.$$  \hspace{1cm} \text{(B.20)}

If (c) holds for $z_j$ then we have

$$\eta(x) \geq r_0(p_i) - 2 \geq \frac{m}{2} - 2.$$  \hspace{1cm} \text{(B.21)}

because the tenant's sample of the $v_a$-memory $z_j(v_a)$ may be a pure $(v_a - x)$ sample as long as $z_j(v_a)$ contains no more than $m/2$ experimental demands. Combining (B.19) to (B.21) with (B.15) to (B.18), (B.2), and (B.3), implies (B.10).

\textit{Proof of (B.11).} We define a path $p = (s_1, \ldots, s_n)$ ($n \geq 2$) with the properties $s_i \in \Theta$ ($i = 1, \ldots, n$), $s_1 = s_{x*}^a$, $s_n = s_{x*+\delta}^a$, and $r_0(p) = m_2(x, x + \delta)$.

Starting from state $s_1 = s_{x*}^a$ at the beginning of some period $t$, the tenants experiment with $(v_a, x + \delta)$ in every period $t, \ldots, t' - 1$ (where $t' = t + m_2(x, x + \delta)$), while the landlords do not experiment. The outcome of these periods is $(v_a, x + \delta, v_a - x)$. From period $t'$ onwards, no experiment occurs. In the periods $t', \ldots, t'' - 1$ (where $t'' = t' + m/2$), the landlords' sample of the $v_a$-memory always contains the demands from periods $t, \ldots, t' - 1$, plus any other demands, and the tenants' sample contains no demands made after period $t' - 1$. The outcome of these periods is $(v_a, x, v_a - x - \delta)$. In the periods $t = t'', \ldots, t'' + m - 1$, the landlords' sample of the $v_a$-memory consists of the $t'' - t''$ most recent demands, plus—if $t - t'' < k$—the demands made in the periods $t'' - (k - (t'' - t'))$ to $t' - 1$, while the tenants' sample of the $v_a$-memory consists of the $k$ most recent demands. The outcome of these periods is $(v_a, x + \delta, x - \delta)$. For $i = 2, \ldots, n$ ($n = t'' + m - t' + 1$), let $s_i$ be the state at the beginning of period $i + t - 1$.

The critical point in this construction is that at the beginning of period $t''$, the period-$t$ demands are still in the $v_a$-memory (because $t'' - m = t'' - m/2 \leq t$, due to $k \leq m/2$). Therefore, in the periods $t'', \ldots, t'' + m - 1$ the landlords' samples are pure $(x + \delta)$-samples and the tenants' samples are pure $(x - \delta)$-samples, implying $s_n = s_{x*+\delta}^a$. Hence, $p$ has the desired properties.
The existence of $p$ implies $r_2(C(x), C(x + \delta)) \leq m_2(x, x + \delta) - 2$. Therefore,

$$\eta(x) \leq m_2(x, x + \delta) - 2 = \tilde{m}_2(x) - 2. \quad (B.22)$$

Proof of (B.12). From (B.1), (B.4), (B.5), and (B.10), we get $\eta(x) \geq \tilde{m}_2(x) - 2$. So, $\eta(x) = \tilde{m}_2(x) - 2$ from (B.11). The claim $C(x + \delta) \in \mathcal{A}(x)$ now follows from the construction of $p$ in the proof of (B.11).

The proofs of (B.13) and (B.14) are analogous to those of (B.11) and (B.12), the only difference being that instead of tenants’ experiments one uses landlords’ experiments with the demand $v_* - x + \delta$ in the $v_*$-subgame.

**Lemma B.9.** We have \( \emptyset \neq \mathcal{A} \cap \mathcal{F} \subseteq \{ \hat{y} : \hat{y} \neq \hat{y} + \delta \} \).

**Proof.** Because \( \mathcal{A} \neq \emptyset \), it is sufficient to show (i) if $\hat{y} = \hat{x} \neq v_* - \delta$ then $\mathcal{A} \subseteq \{ \hat{y}, \hat{y} + \delta \} \subseteq \mathcal{F}$; (ii) otherwise \( \mathcal{A} \subseteq \{ \hat{y} \} \subseteq \mathcal{F} \).

Proof of (i). From $\hat{x} \leq v_* / 2$ and (B.12) we get $\eta(\hat{y}) = \tilde{m}_2(\hat{y}) - 2$. For $x < \hat{y}$, we have $\eta(x) \leq \tilde{m}_2(x) - 2 < \tilde{m}_2(\hat{y}) - 2$ by (B.11) and (B.8). Therefore,

$$\forall x < \hat{y} : x \notin \mathcal{A}. \quad$$

Similarly using $\hat{x} + \delta \geq v_* / 2$, (B.14), (B.13) and (B.9) we get

$$x > \hat{y} + \delta \Rightarrow x \notin \mathcal{A}. \quad$$

Hence, $\mathcal{A} \subseteq \{ \hat{y}, \hat{y} + \delta \}$. From (B.12) and (B.14) we get $\{ \hat{y}, \hat{y} + \delta \} \subseteq \mathcal{F}$.

Proof of (ii). In the case $\hat{y} = \hat{x} = v_* - \delta$ the proof is analogous to that of (i). In the remaining case $\hat{y} \neq \hat{x}$ we have $\tilde{y} = x + \delta$. From $\tilde{y} \geq v_* / 2$, (B.1), (B.9), and (B.8) we get $m_1(\tilde{y}) \leq \tilde{m}_2(\tilde{y})$. Hence, $\eta(\tilde{y}) \geq \tilde{m}_2(\tilde{y}) - 2$ by (B.10). On the other hand, for all $x > \tilde{y}$, we have $\eta(x) \leq m_1(x) - 2 < m_1(\tilde{y}) - 2$ by (B.13) and (B.9). Therefore, $\mathcal{A} \subseteq \{ \tilde{y} \}$. Finally, (B.14) implies $\tilde{y} \in \mathcal{F}$.

To complete the proof of Proposition 4.1, combine Proposition A.1 ($i = 1$) with Lemmata B.7, B.9, and B.6.

**REFERENCES**

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